P47 Optics

Mathematics for Optics

- ★ multivariable, vector calculus (Math 13)
- linear algebra (math 22/24)
- Fourier theory
- complex-valued functions

(Fourier theory and complex-valued functions: Engs 92...)

[linear algebra, Fourier theory, complex-valued functions:] for these, you'll learn as needed during the course of the term

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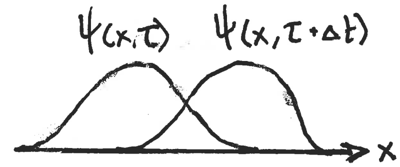
Review of Basic Wave Mechanics

Wave: traveling, self-sustaining disturbance of some "field"

- described by a function that depends on space and time in a very specific way:

$$\psi(x,t) = f(x \pm vt) \quad \text{[1D wave]}$$



- direction of propagation: $-vt$ goes toward $+x$
  (perhaps opposite your intuition?) $+vt$ goes toward $-x$

functions like this satisfy the 1D scalar wavefunction:

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

[Note: linear, homogeneous 2nd order partial differential equation]

In 3D:

$$\nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}, \quad \text{where } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \text{("Laplacian" operator)}$$

"Harmonic" waves are the simplest solution (lots of ways to write them)

Ex: $\psi(x, t=0) = A \sin\left(\frac{2\pi x}{\lambda}\right)$, λ: wavelength more generally:

$$\Psi(x,t) = A\sin(kx - \omega t + \varphi_0)$$

(Note: if $\varphi_0 = \pi/2$, then sin → cos)

$k = 2\pi/\lambda$: wavenumber
$\omega = 2\pi\nu$: angular frequency
ν : temporal frequency
φ_0 : initial phase

the whole argument can be defined as "the phase"

$$\varphi(x,t) = kx - \omega t + \varphi_0$$

Phase Velocity

$$\left|\left(\frac{\partial\varphi}{\partial t}\right)_x\right| = \omega \quad \& \quad \left|\left(\frac{\partial\varphi}{\partial x}\right)_t\right| = k$$

So the velocity of a monochromatic wave is given by

$$\left(\frac{\partial x}{\partial t}\right)_\varphi = \frac{-(\partial\varphi/\partial t)_x}{(\partial\varphi/\partial x)_t} = \pm\frac{\omega}{k} = V_\varphi$$

Note: the velocity of wave packets is not so simple as this.
Later, we'll discuss the idea of "group" velocity: $V_g = \frac{d\omega}{dk}\Big|_{\bar{\omega}}$
This is because, in general, we'll be dealing with <u>superpositions</u> of waves at different frequencies, where the frequency components travel at different velocities due to <u>dispersion</u>.

$\Longrightarrow$ in general: V_φ depends on ω!

Complex Reprensentation of Harmonic Functions

Sums and products of trig functions can be algebraically messy

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

There's a mathematically simpler way to represent harmonic waves

Euler's Formula: $e^{i\phi} = \cos\phi + i\sin\phi$

Exponential Representation of trig functions:

$$\cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2} = \mathrm{Re}(e^{i\phi})$$

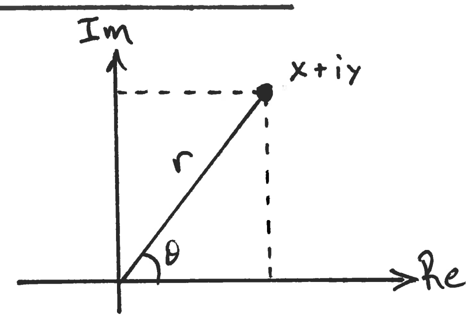
$$\sin\phi = \frac{e^{i\phi} - e^{-i\phi}}{2i} = \mathrm{Im}(e^{i\phi})$$

Connection to hyperbolic trig: $(\phi = i\gamma)$

$$\cos(i\gamma) = \frac{e^{-\gamma} + e^{\gamma}}{2} = \cosh(\gamma)$$

$$\sin(i\gamma) = \frac{e^{-\gamma} - e^{\gamma}}{2i} = i\sinh(\gamma)$$

Quick Review of Complex Numbers:



$$\tilde{z} = a + ib = re^{i\theta}$$

Cartesian - cylindrical connection:

$a = r\cos\theta$

$b = r\sin\theta$

$r = \sqrt{a^2 + b^2}$

$\theta = \arctan(b/a)$

$\tilde{z}^* = a - ib = re^{-i\phi}$ $\Longrightarrow$ $\mathrm{Re}(\tilde{z}) = \frac{1}{2}(\tilde{z} + \tilde{z}^*) = a$, $\mathrm{Im}(\tilde{z}) = \frac{1}{2i}(\tilde{z} - \tilde{z}^*) = b$

We're often interested in waves of the form:

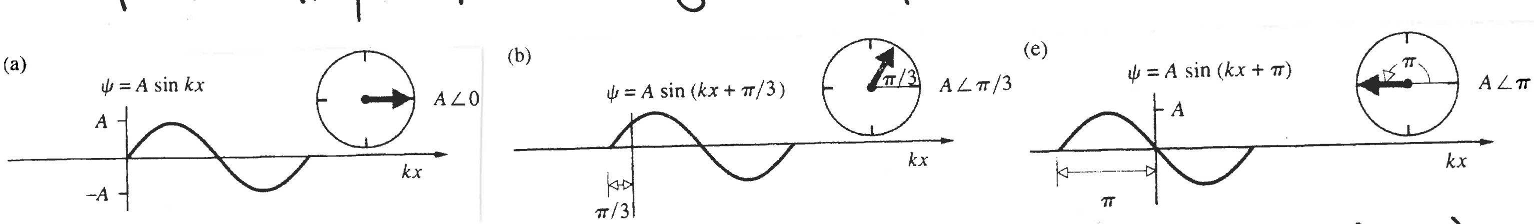
$$\psi(x,t) = A\cos(\omega t - kx + \varphi_0) = \mathrm{Re}\left(Ae^{i(\omega t - kx + \varphi_0)}\right)$$

This occurs so often, it's customary to drop the Re( ) as a shorthand

$$\psi(x,t) = Ae^{i(\omega t - kx + \varphi_0)} = Ae^{i\phi}$$

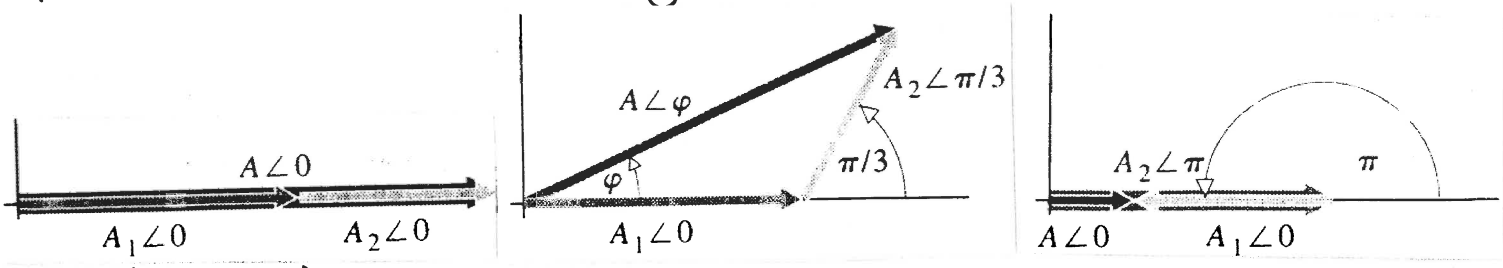
Phasor Representation:

"positive" frequency vs. "negative" frequency



(180 degrees out of phase)

Addition of phasors representing 2 waves:



(in phase) (partly out of phase) (out of phase)

We can use this to add multiple waves of different amplitude and phase

$$\Rightarrow \tilde{z}_1 + \tilde{z}_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

A note on sign conventions:

Should we use $\cos(kx - \omega t)$ or $\cos(-kx + \omega t)$?

Here, it doesn't really matter since cos is an even function, but...

$$\sin(kx - \omega t) = -\sin(-kx + \omega t)$$

So, if we're using exponentials

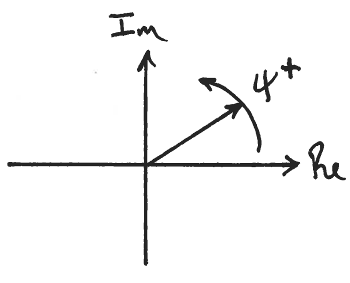
$$\psi^+ = e^{i(kx-\omega t)} = \cos(\cdots) + i\sin(\cdots)$$

$$\psi^- = e^{-i(kx-\omega t)} = \cos(\cdots) - i\sin(\cdots)$$

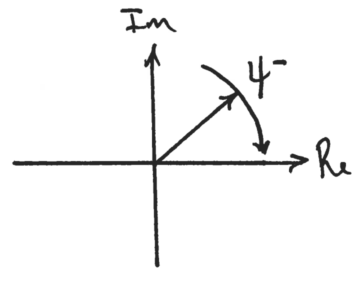
does this even matter? (not if it's used consistently)

Nearly all references use the ψ^+ "right-handed" convention (although, "negative" frequency components are also needed in Fourier analysis... later)

Another way to think about this is as a "coordinate" degree of freedom by looking at the phasor representation



Phasor rotates CCW as time advances



Phasor rotates CW as time advances

Plane Waves

Generalizing a 1D wave to 3D space is fairly simple

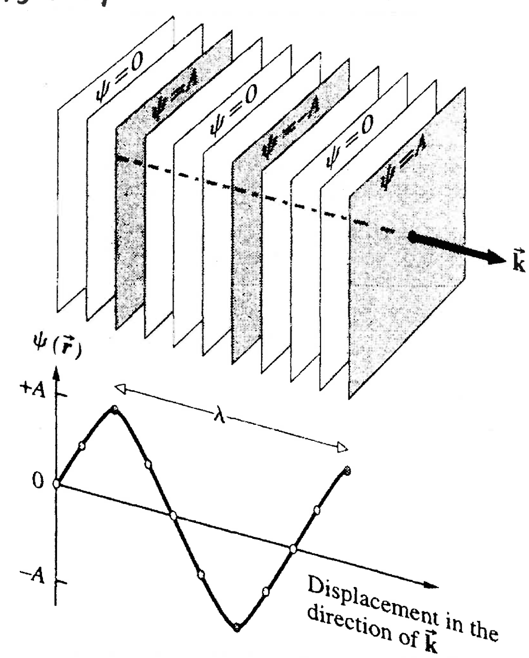
(1D) $\psi(x,t) = Ae^{i(kr-\omega t)}$

(3D) $\psi(x,y,z,t) = Ae^{i(k_x x + k_y y + k_z z - \omega t)}$

(coordinate independent form) $\psi(\vec{r},t) = Ae^{i(\vec{k}\cdot\vec{r} - \omega t)}$ | cartesian vector: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
propagation vector: $\vec{k} = k_x\hat{i} + k_y\hat{j} + k_z\hat{k}$

Note: $|\vec{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2} = k$
(the same (scalar) propagation const. from the 1D case) (be careful! unfortunately 'k' gets recycled far too much.)

$\vec{k}\cdot\vec{r}$ = const. which defines a planar surface perpendicular to $\vec{k}$ – a "surface of equal phase" or "wavefront"



Can already see that for one plane wave, can rotate coordinates to align with direction of propagation, making the problem 1D

But, if we have 2+ plane waves propagating in different directions, we'll need to keep track

In general, though, any 3D wave can be reduced to a sum of plane waves (superposition principle)

3D Wave Equation (a.k.a. Helmholtz Eqn.) (scalar version)

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

$$\Rightarrow \nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$ (coord. independent form)

∇^2: Laplacian Operator

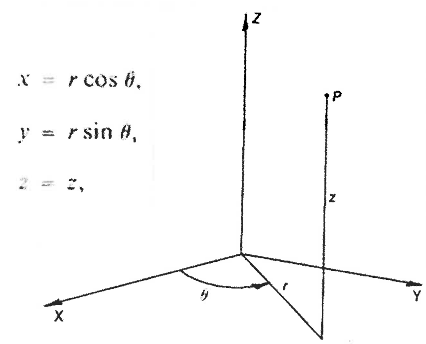
[Side Note:
Here, ψ is a scalar, so $\nabla^2 \psi = \nabla \cdot \nabla \psi$, which is fine for now
However, if a vector, then: $\nabla^2 \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla \times (\nabla \times \vec{E})$
but still straight forward in Cartesian: $\nabla^2 \vec{E} = \nabla^2 E_x \hat{i} + \nabla^2 E_y \hat{j} + \nabla^2 E_z \hat{k}$]

Validate any plane wave expression by plugging into wave eqn. as a check

Ex: $\nabla^2 \psi(\vec{r},t) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) A e^{i(k_x x + k_y y + k_z z - \omega t)}$

$= -A(k_x^2 + k_y^2 + k_z^2) e^{i(\cdots)} = -k^2 \psi(\vec{r},t)$

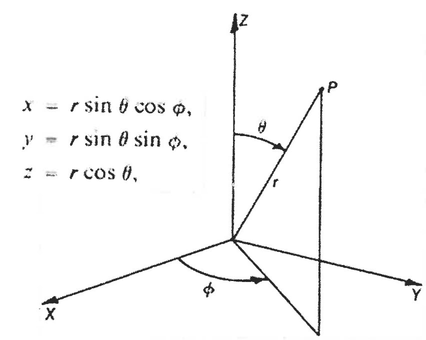
$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi(\vec{r},t) = -\frac{\omega^2}{v^2}\psi(\vec{r},t) = -k^2 \psi(\vec{r},t)$ $=$ ✓



Problems don't always have cartesian symmetry

Cylindrical: $\psi(\vec{r}) = \psi(r, \theta, z)$

spherical: $\psi(\vec{r}) = \psi(r, \theta, \phi)$



What if we're only concerned with purely radial waves?

Cylindrical: drop θ, z dependence

spherical: drop θ, ϕ dependence

The spherical Laplacian can be rewritten to give a much simpler "effective 1D" wave equation in spherical coordinates

$$\frac{\partial^2}{\partial r^2}(r\psi) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}(r\psi)$$

"Harmonic Spherical Wave" - a special case of this general radial wave eqn. solution

$\psi(r,t) = \frac{A}{r} e^{i(kr \mp \omega t)}$ | $-$ is outgoing, $+$ is incoming

here, the $\frac{1}{r}$ is necessary for conservation of energy as wave moves outward

EM Waves in Free Space

EM concept review:

Gauss' Law ($\vec{E}$): $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \rightarrow 0$ (no charges in vacuum) $\quad \oiint_A \vec{E} \cdot d\vec{S} = 0$ (charges source E-fields)

Gauss' Law ($\vec{B}$): $\vec{\nabla} \cdot \vec{B} = 0$ (no magnetic "charges" to begin with) $\quad \oiint_A \vec{B} \cdot d\vec{S} = 0$ (monopoles source B-fields)

[differential form] [integral form]

Faraday's Law: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (changing B makes E)

Ampere's Law: $\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J} \rightarrow 0$ (no currents) (changing E makes B)

(Note: $\mu_0 \epsilon_0 = 1/c^2$)

How to get the wave equations? Faraday & Ampere Laws look close - need 2nd deriv.

take curl of Faraday's Law

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E})^{\rightarrow 0} - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

$$-\nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

take curl of Ampere's Law

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \times \left(-\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{B})^{\rightarrow 0} - \nabla^2 \vec{B} = -\mu_0 \epsilon_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E})$$

$$-\nabla^2 \vec{B} = -\mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(\frac{\partial \vec{B}}{\partial t}\right)$$

$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$$

Two coupled fields: $\quad \vec{E}(\vec{r}, t) \qquad \vec{B}(\vec{r}, t)$

Now, assume independent solutions to the $\vec{E}$ & $\vec{B}$ wave equations:

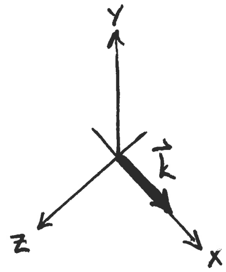
$$\vec{E}(\vec{r}, t) = \vec{E}_0 \cos(\vec{k}_E \cdot \vec{r} - \omega_E t + \phi_E)$$

$$B(\vec{r}, t) = \vec{B}_0 \cos(\vec{k}_B \cdot \vec{r} - \omega_B t + \phi_B)$$

[Note: $\vec{E}_0$ and $\vec{B}_0$ are vectors, not scalars. More on polarization later.]

All solutions of Maxwell's equations are solutions of the vector wave equation, but not visa vers

Given this, what extra constraints on the solutions do we have?



to simplify, set $\vec{k}_E = k\hat{\imath}$, $\omega_E = \omega$, $\phi_E = 0$

$$\Rightarrow \vec{E}(\vec{r},t) = \vec{E}_0 \cos(kx - \omega t)$$

I) Gauss' Law

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \Rightarrow \frac{\partial E_x}{\partial x} = 0 \quad \text{(no y or z dependence)}$$

- if there's <u>any</u> field in the $\hat{\imath}$ (propagation) direction, it <u>has</u> to be constant, i.e. <u>not a wave</u>
- so, $\vec{E}_0$ <u>has</u> to be oriented in the y-z plane (transverse wave)
- implies the <u>same constraint on $\vec{B}$.</u>

II) Faraday's Law

- again, simplify by choosing $\vec{E}_0 = E_y \hat{\jmath}$ (more on polarization later)

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \frac{\partial E_y}{\partial x} - \cancel{\frac{\partial E_x}{\partial y}}^{\,0} = -\frac{\partial B_z}{\partial t}$$

$$\Rightarrow \frac{\partial}{\partial x} E_y \cos(kx - \omega t) = -\frac{\partial}{\partial t} B_z \cos(\vec{k}_B \cdot \vec{r} - \omega_B t + \phi_B)$$

$$\Rightarrow -kE_y \sin(kx - \omega t) = -\omega_B B_z \sin(\vec{k}_B \cdot \vec{r} - \omega_B t + \phi_B)$$

- only way to satisfy this is $\vec{k}_B = \vec{k}_E = k\hat{\imath}$, $\omega_B = \omega_E = \omega$, $\phi_B = \phi_E = 0$

$$\Rightarrow kE_y \cancel{\sin(kx - \omega t)} = \omega B_z \cancel{\sin(kx - \omega t)}$$

$$\Rightarrow E_y = \frac{\omega}{k} B_z = cB_z$$

Summary)

- $\vec{E}_0$ and $\vec{B}_0$ are transverse ($\perp$ to $\vec{k}$)
- same direction, frequency, wavelength
- in phase with each other
- perpendicular to each other
- $|E| = c|B|$

Energy and Momentum of EM Waves

E and B fields contain stored energy density

$u_E = \frac{\epsilon_0}{2} E^2$ (capacitor) & $u_B = \frac{1}{2\mu_0} B^2$ (solenoid) $\quad \frac{1}{c^2} = \mu_0 \epsilon_0$

and since $E = cB$, then $u_B = \frac{1}{2\mu_0} B^2 = \frac{1}{2\mu_0} (E/c)^2 = \frac{1}{2\mu_0} E^2 \mu_0 \epsilon_0 = \frac{\epsilon_0}{2} E^2 = u_E$

- the E and B fields of an EM wave store equal amounts of energy!
- the total energy density can be written in terms of E or B

$$u = u_E + u_B = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 = \epsilon_0 E^2 = \frac{1}{\mu_0} B^2$$

Now, consider transport of energy EM waves

the wave with energy density $u = \epsilon_0 E^2$ is traveling at velocity c

the energy passing a given area per unit time is simply

$$S = uc = \epsilon_0 E^2 c \quad \left(= \epsilon_0 E B c^2 = \frac{1}{\mu_0} E B \right)$$ ← lots of ways to rewrite this

energy ought to flow in the direction of wave propagation, so

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = c^2 \epsilon_0 \vec{E} \times \vec{B} \quad (\text{since } \hat{E} \times \hat{B} = \hat{k})$$ $\quad \left(\text{also } = \epsilon_0 E^2 \vec{k} = \frac{1}{\mu_0} B^2 \vec{k}\right)$

the vector $\vec{S}$ here is called the Poynting vector

How much power is flowing at any particular time?

$$\vec{S} = c^2 \epsilon_0 \vec{E}_0 \times \vec{B}_0 \cos^2(\vec{k} \cdot \vec{r} - \omega t)$$ ("standard" harmonic wave)

See units are (energy)/(area · time), but delivery of energy is not constant

Averaging Harmonic Functions:

- ω is usually much too fast for us to detect the variation, detectors only measure the average power over some window in time

Time average of any function over an interval T, centered around time t

$$\langle f(t)\rangle_T = \frac{1}{T}\int_{t-T/2}^{t+T/2} f(t)\,dt$$

(Note: can choose integration limits $\int_t^{t+T}$ or $\int_{t-T/2}^{t+T/2}$)

time average of harmonic functions

$$\langle \cos\omega t\rangle_{T\to\infty} = \langle \sin\omega t\rangle_{T\to\infty} = 0 \quad , \quad \langle e^{i\omega t}\rangle_{T\to\infty} = 0$$

$$\langle \cos^2\omega t\rangle_{T\to\infty} = \langle \sin^2\omega t\rangle_{T\to\infty} = \frac{1}{2} \quad (!)$$

Okay, but what about $0 < T < \infty$? $\langle f(t)\rangle_T$ depends on T!

$$\langle e^{i\omega t}\rangle_T = \frac{\sin \omega T/2}{\omega T/2}\, e^{i\omega t} = \text{sinc}\left(\frac{\omega T}{2}\right) e^{i\omega t}$$

$$\langle \sin\omega t\rangle_T = \text{sinc}\left(\frac{\omega T}{2}\right)\sin\omega t$$

$$\langle \cos\omega t\rangle_T = \text{sinc}\left(\frac{\omega T}{2}\right)\cos\omega t$$

$$\langle \cos^2\omega t\rangle_T = \frac{1}{2}\left[1 + \text{sinc}(\omega T)\cos(2\omega t)\right]$$

$$\boxed{\text{sinc}\, x = \frac{\sin x}{x}}$$

the "cardinal sine"

EM Momentum & Radiation Pressure

Now, remember energy density?

has units $[u] = \frac{J}{m^3} = \frac{N\cdot m}{m^2\cdot m} = \frac{N}{m^2} = \frac{[F]}{[A]}$ pressure?!

So EM radiation exerts a pressure equal to its energy density

Well, whenever there's energy transfer, there ought to be momentum transfer

$$S = uc \implies P = u = \frac{S}{c} \text{ (pressure)} \quad \& \quad P_V = \frac{S}{c^2} \text{ (momentum per unit vol.)}$$

Or, time averaged $\langle P(t)\rangle_T = \frac{\langle S(t)\rangle_T}{c} = \frac{I}{c}$ ← irradiance

Note: change in momentum of an object depends on absorption, reflection, emissio

Side Note: Quantization

- energy and momentum are <u>actually</u> exchanged in discrete packets

$$E_\gamma = h\nu = \hbar\omega \quad \& \quad p_\gamma = \frac{h}{\lambda} = \hbar k$$

A few additional comments:

Forces on charges from EM waves: Lorentz Force Law: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

How much force on a (moving) charge due to an EM wave?

$$F = q\left(\vec{E} + \frac{\vec{v} \times \vec{E}}{c}\right)$$

- $\vec{E}(\vec{r},t)$ needs to be somewhat large for first term to be significant
- for second term to become significant, either $\vec{E}(\vec{r},t)$ needs to be <u>very</u> large, or the particle be moving <u>very</u> fast