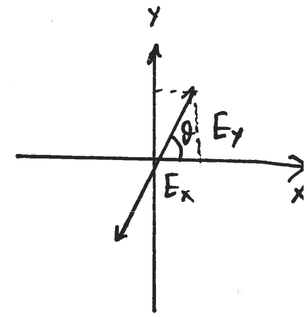
Polarization

Back in Unit 2, briefly talked about the fact that E-field is a vector and could consider a general EM wave as being composed of two orthogonal waves with arb. difference in overall phase

$$E(z,t) = E_x \cos(kz - \omega t)\,\hat{i} + E_y \cos(kz - \omega t - \phi_0)\,\hat{j}$$

- if $\phi_0 = 0$, have <u>linear polarization</u>



$$E(z,t) = (E_x\hat{i} + E_y\hat{j})\cos(kz - \omega t)$$

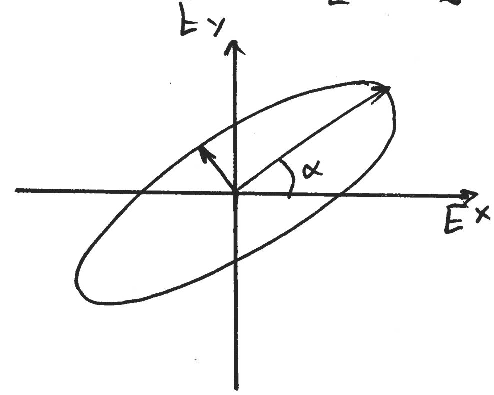
$$\Rightarrow E_0 = \sqrt{E_x^2 + E_y^2} \quad \& \quad \theta = \tan^{-1}\left(E_y/E_x\right)$$

- but if $E_x = E_y = E_0$ <u>AND</u> $\phi_0 = \pm\pi/2$, have <u>circular polarization</u>

- You know by now that EM waves carry linear momentum, but they also carry angular momentum
  - an object absorbing circularly polarized light feels the torque
  - optics which change the angular momentum of the transmitted beam experience a torque

  - photons are spin-1 bosons ($\pm\hbar$ of angular momentum)

- the general state of polarization is an ellipse



Parametric representation of an ellipse:

$$E_x = E_{ox}\cos(t+\phi_x)$$
$$E_y = E_{oy}\cos(t+\phi_y)$$

- for shape of ellipse, only relative phase matters

$$\phi_y - \phi_x = \delta$$

- for effects like interference, though, only the overall (absolute) phase matters

- combine equations, do algebra ⟶ eliminate time dependence

$$\left(\frac{E_x}{E_{ox}}\right)^2 + \left(\frac{E_y}{E_{oy}}\right)^2 - 2\left(\frac{E_x E_y}{E_{ox}E_{oy}}\right)\cos\delta = \sin^2\delta$$

and $\tan 2\alpha = \dfrac{2E_{ox}E_{oy}\cos\delta}{E_{ox}^2 - E_{oy}^2}$

which gives the equation for an ellipse that is rotated by angle α relative to the E_x axis

- The above is okay for illustration purposes, but this type of representation isn't very convenient/practical for calculations.

<u>Jones Matrix Representation of Pure Polarization States</u> (see Hecht 8.13)

(for partially polarized light, see Mueller matrices and Stokes parameters)

- given a monochromatic wave of arb. polarization state:

$$\vec{E}(z,t) = E_{ox}\, e^{i(kz-\omega t+\phi_x)}\,\hat{i} + E_{oy}\, e^{i(kz-\omega t+\phi_y)}\,\hat{j}$$

- First, make use of the fact that can represent the Amplitude & phase together as a complex number

$$\vec{E}(z,t) = \left(E_{ox}\, e^{i\phi_x}\,\hat{i} + E_{oy}\, e^{i\phi_y}\,\hat{j}\right) e^{i(kz-\omega t)}$$

$$= E_{eff}\left(A\hat{i} + B e^{i\delta}\hat{j}\right) e^{i(kz-\omega t)}$$

where we've defined an overall field amplitude: $E_{eff} = e^{i\phi_x}\sqrt{|E_{ox}|^2 + |E_{oy}|^2}$

and have numbers describing the state of each component:

$$A = \frac{|E_{ox}|}{\sqrt{|E_{ox}|^2 + |E_{oy}|^2}} \quad \& \quad B = \frac{|E_{oy}|}{\sqrt{|E_{ox}|^2 + |E_{oy}|^2}}$$

- Here, A & B are non-negative, real, dimensionless numbers and satisfy:

$$A^2 + B^2 = 1$$

<u>Note:</u> if A or B = 0, then the phase of that component is considered indeterminant and we only consider the phase of the non-zero component valid

- At this point, can write the state of the EM field as

$$\vec{E}(z,t) = E_{eff}\begin{bmatrix} A \\ Be^{i\delta} \end{bmatrix} e^{i(kz-\omega t)}$$

- In many problems involving polarization, aren't concerned with the oscilliatory part – so just drop it (understand: it's still there!)

$$\vec{E}(z,t) = E_{eff}\begin{bmatrix} A \\ Be^{i\delta} \end{bmatrix} = E_{eff}\,\hat{\epsilon}$$, where $\hat{\epsilon}$ is Jones Vector

- Of course, normalization still holds, so:

$$\hat{\epsilon}^* \cdot \hat{\epsilon} = \begin{bmatrix} A & Be^{i\delta} \end{bmatrix}\begin{bmatrix} A \\ Be^{i\delta} \end{bmatrix} = A^2 + B^2 = 1$$

- What are the matrix representations of various polarization states?

$$\left.\begin{array}{l} \text{1) \underline{Linearly polarized along } \hat{x}\text{}:} \quad \epsilon_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \\ \text{2) \underline{Linearly polarized along } \hat{y}\text{}:} \quad \epsilon_y \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{array}\right\} \text{linear basis states}$$

3) <u>Linearly polarized at angle α w.r.t. x</u>:

$$R(\alpha)\,\hat{\epsilon}_x = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

<u>Note:</u> All these vectors have an overall phase degree of freedom within E_{eff}

$$\hat{\epsilon} = \begin{bmatrix} i \\ 0 \end{bmatrix} = \begin{bmatrix} e^{i\pi/2} \\ 0 \end{bmatrix} = e^{i\pi/2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ still horizontally polarized in $\hat{x}$! But $\phi_x = \pi/2$.

4) Circular Polarized light:

$$\hat{\epsilon}_L = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\end{bmatrix} \quad \text{"positive helicity"} \quad \hat{\epsilon}_+$$

$$\hat{\epsilon}_R = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -i\end{bmatrix} \quad \text{"negative helicity"} \quad \hat{\epsilon}_-$$

} circular basis vectors

- the motivation for using this formalism is that we can represent any linear optical element's effect on the polarization with a complex-valued 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A' \\ B' \end{bmatrix} \quad \text{or} \quad M\hat{\epsilon} = \hat{\epsilon}'$$

- and if we have a succession of elements → just chain the matrices

$$\hat{\epsilon}' = M_N \cdot M_{N-1} \cdots M_2 M_1 \hat{\epsilon}$$

where the matrices are non-commutative ⟶ order matters!

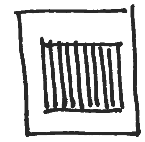
Polarization Optics

Polarization is affected in many ways when light interacts with a material

- reflections from surfaces (Fresnel)
- Scattering from very small particles (the sky is polarized!)
- Non-isotropic physical materials:
  - polarization-dependent absorption
  - polarization-dependent index of refraction
    ⟶ leads to polarization-dependent phase shifts

Polarizers

- basically anything that allows one polarization component to pass through, but disallows the orthogonal component
- ~~Ex:~~ Wire-grid polarizer: allows $\vec{E} \perp$ to wires, forbids $\parallel$
  (Transmission Axis is in $\leftrightarrow$ direction)



- How would we represent, mathematically, the action of a polarizer on an arbitrary input polarization?
- Take the wire grid array shown above with a horizontal transmission axis

$$P_H = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow P_H \cdot \hat{\epsilon} = \hat{\epsilon}_x \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} A \\ Be^{i\delta} \end{bmatrix} = \begin{bmatrix} A \\ 0 \end{bmatrix}$$

What elements are needed? Obviously: $P_H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; likewise $P_V = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

- What if we rotate the polarizer by an arbitrary angle?

Remember: • to rotate a 1st rank tensor (vector): $R \cdot \vec{v}$
• to rotate a 2nd rank tensor (matrix): $R \cdot M \cdot R^T$

Here, $R = R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

and $R^T(\theta) = R^{-1}(\theta) = R(-\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$P_\theta = R(\theta) P_H R(-\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

Malus' Law

For linearly polarized light (in arb. direction) incident on a linear polarizer, how does the transmitted intensity change as a function of angle?

$$E(\theta) = P_H \cdot R(\theta) \cdot \vec{E} = E_{eff} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = E_{eff} \begin{bmatrix} \cos\theta \\ 0 \end{bmatrix}$$

$$I \propto E^2 \implies \underline{\underline{I(\theta) = I_0 \cos^2\theta}}$$

Cascaded Polarizers

What happens when you apply successive polarizers to unpolarized beam?

<u>Case 1</u>: $E(\theta) = P_H \cdot P_V \cdot \vec{E} = E_{eff}\, P_H \cdot P_V \cdot \hat{\mathcal{E}}_{in} = E_{eff}\, \hat{\mathcal{E}}_{out}$

$$\Rightarrow E(\theta) = E_{eff} \cancelto{0}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} A \\ B \end{bmatrix} = E_{eff} \begin{bmatrix} A' \\ B' \end{bmatrix}, \text{ so } A' = B' = 0.$$

<u>Case 2</u>: $E(\theta) = P_H \cdot P_\theta \cdot P_V \cdot \vec{E} = E_{eff} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$

$$= E_{eff} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} 0 \\ B \end{bmatrix}$$

$$= E_{eff} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B\cos\theta\sin\theta \\ B\sin^2\theta \end{bmatrix}$$

$$= E_{eff} \begin{bmatrix} B\cos\theta\sin\theta \\ 0 \end{bmatrix}$$

$$= E_{eff} \begin{bmatrix} A' \\ B' \end{bmatrix}, \text{ so } A' = B\cos\theta\sin\theta \;\&\; B' = 0 \; !$$

Case 3: $E(\theta) = P_H \cdot P_{N\delta\theta} \cdot P_{(N-1)\delta\theta} \cdots P_{\delta\theta} P_V \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A' \\ B' \end{bmatrix}$ [5-8]

- Can simply stack N polarizers (Some materials act like this → twisted nematic liquid crystals)
- when $N \to \infty$ and $\delta\theta \to 0$,

  then $\begin{bmatrix} A' \\ B' \end{bmatrix} \to \begin{bmatrix} 0 \\ A \end{bmatrix}$, complete polarization rotation w/ marginal loss

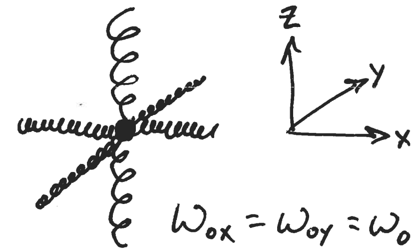
  (an optical element known as a wave retarder)

Birefringence

- Only example of a polarizing element given so far is the wire grid, but some crystals/polymers also have this optical property.

- How do they do that?

  → birefringence: index of refraction determined by orientation of material's internal structure



$\omega_{0x} = \omega_{0y} = \omega_0$

$\omega_{0z} = \omega_e > \omega_0$

Ex: the "uniaxial" crystal: calcite (biaxial also exists, where have 3 different n)

→ (meaning, one internal optical axis)

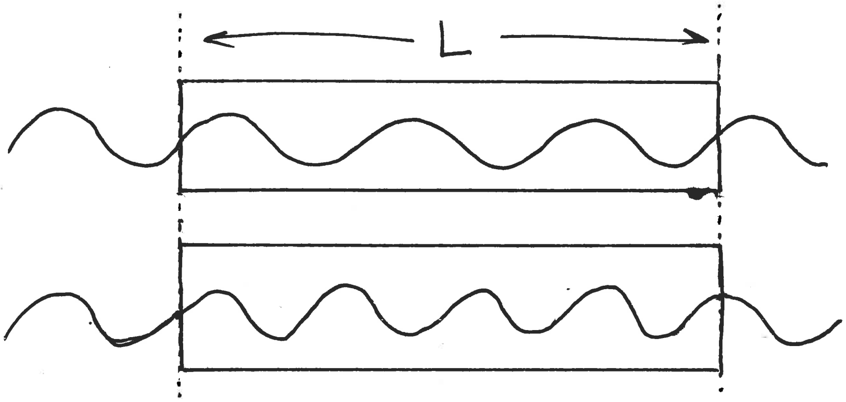
Case 1: beam propagating along $\hat{z}$

- if $\hat{\epsilon} = \hat{x}$ or $\hat{\epsilon} = \hat{y}$, then $n = n_0$

Case 2: beam propagating along $\hat{x}$

- if $\hat{\epsilon} = \hat{y}$, then $n = n_0$ ("ordinary")
- if $\hat{\epsilon} = z$, then $n = n_e$ ("extraordinary")

- there is <u>a lot</u> we could do with birefringent crystals, but we'll keep it fairly simple
- to conceptualize this, imagine light passing through a block of length L of two different materials: one with $n = n_o$ and other with $n = n_e$



$$\phi_o(L) = \frac{2\pi L}{\lambda_{ord.}} = \frac{2\pi n_o L}{\lambda_{vac}}$$

$$\phi_e(L) = \frac{2\pi L}{\lambda_e} = \frac{2\pi n_e L}{\lambda_{vac}}$$

- the exiting beams will have a relative phase difference: $\underline{\underline{\Delta\phi = \frac{2\pi(n_e - n_o)}{\lambda_{vac}} L}}$
- degree of <u>birefringence</u> is determined by $\Delta n = n_e - n_o$
- can see that, for a beam with both pol. components (eg: linear @ 45° w.r.t. $\hat{z}$), then the two polarization components experience a relative phase shift

~~Ex:~~ How far would 550 nm light have to travel in calcite and in quartz for the two polarization components to become 180° out-of-phase?

<u>Calcite (strong birefringence)</u>

$n_o = 1.65838$ "slow" axis
$n_e = 1.48643$ "fast" axis
$\}$ at 550nm

$$L_\pi = \frac{1}{2}\frac{\lambda_{vac}}{|n_e - n_o|} = 1.6\,\mu m$$

<u>Quartz (weak birefringence)</u>

$n_o = 1.54422$ "fast" axis
$n_e = 1.55332$ "slow" axis
$\}$ at 550 nm

$$L_\pi = \frac{1}{2}\frac{\lambda_{vac}}{n_e - n_o} = 30.2\,\mu m$$

<u>Note:</u> How do you go about manufacturing/cleaving a 1" diameter disk at this thickness?

Wave Retarders

Optical elements which transform the polarization state of beam (variously known as <u>retarders</u>, <u>wave plates</u>, <u>rotators</u>, and <u>compensators</u>)

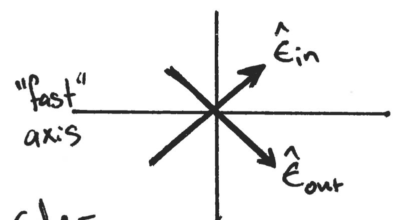
Jones Matrix Representation: $L = \begin{bmatrix} e^{i\phi_o} & 0 \\ 0 & e^{i\phi_e} \end{bmatrix} = e^{i\phi_o}\begin{bmatrix} 1 & 0 \\ 0 & e^{i\Delta\phi} \end{bmatrix}$

<u>Case 1:</u> retarder with $\Delta\phi = \pi$ acting on linearly pol. beam (45°)

linearly polarized at 45° means $\hat{\mathcal{E}}_{45} = \begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

so $L(\Delta\phi = \pi)\cdot\hat{\mathcal{E}}_{45} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\longrightarrow$ Polarization rotated by 90° !



Such an optical element is called a <u>$\frac{\lambda}{2}$-plate/retarder</u>

<u>Case 2:</u> arb. orientation of $\frac{\lambda}{2}$-retarder acting on horiz. polarized light

first, $R(\theta)L(\pi)R(-\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

then $R(\theta)L(\pi)R(-\theta)\cdot\hat{\mathcal{E}}_x = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos 2\theta \\ \sin 2\theta \end{bmatrix}$

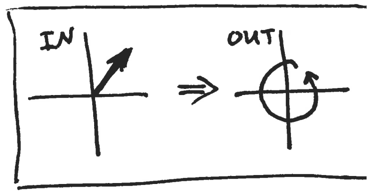
$\longrightarrow$ Polarization gets rotated by 2θ !

(this is a <u>very</u> useful optical device)

Case 3: (Same as Case #1, but $\Delta\phi = \pi/2$ instead)

$$L(\pi/2)\cdot\hat{\epsilon}_{45} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & e^{i\pi/2}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$$

$\longrightarrow$ linear polarization converted to circular!



These optical elements are called "Quarter-wave plates" or "$\lambda/4$-plate" or "$\lambda/4$-retarder"

Case #4: (same as Case #2, but with the $\frac{\lambda}{4}$-plate instead)

Arbitrary orientation of $\frac{\lambda}{4}$-plate:

$$R(\theta)L(\pi/2)R(-\theta) = \begin{bmatrix}\cos^2\theta + i\sin^2\theta & (1-i)\sin\theta\cos\theta\\ (1-i)\sin\theta\cos\theta & \sin^2\theta - i\cos^2\theta\end{bmatrix}$$

$\longrightarrow$ allows general conversion to/from elliptically polarized states

Practical Wave Retarders

We saw above that a quartz $\lambda/2$ ($\Delta\phi = \pi$) retarder needs to be 30.2 μm thick (for 550 nm light)

Note: dispersion for n_o & n_e are both wavelength dependent, and so $\Delta\phi$ also depends on wavelength

- it's very difficult to make a large-diameter, flat, 30 μm thick quartz sheet
- instead, usually employ multi-order wave retarders, where thickness results in an overall phase shift of $\Delta\phi = 2\pi m + \varepsilon$ (where $\varepsilon = \pi, \pi/2$, etc)

Ex: What is the thickness difference for a $\frac{\lambda}{2}$- and $\frac{\lambda}{4}$-plate made of quartz, but having $m = 50$ orders? ($\lambda = 671\,nm$)

(typical high-powe DPSS Laser (45

$$\Delta\phi_{\pi} = \pi + 2\pi(50) = 101\pi$$

$$\Delta\phi_{\pi/2} = \frac{\pi}{2} + 2\pi(50) = 100.5\pi$$

$$L_{\pi} = \frac{\Delta\phi_{\pi}}{2\pi}\cdot\frac{\lambda}{\delta n} = \frac{(101\pi)}{2\pi}\cdot\frac{(671\,nm)}{(0.00901)} = 3.761\ mm$$

$$L_{\pi/2} = \frac{\Delta\phi_{\pi/2}}{2\pi}\cdot\frac{\lambda}{\delta n} = \frac{(100.5\pi)}{2\pi}\cdot\frac{(671\,nm)}{(0.00901)} = 3.742\ mm$$

$$\Longrightarrow \Delta L = \underline{\underline{18.6\,\mu m}}$$

Circular Dichroism

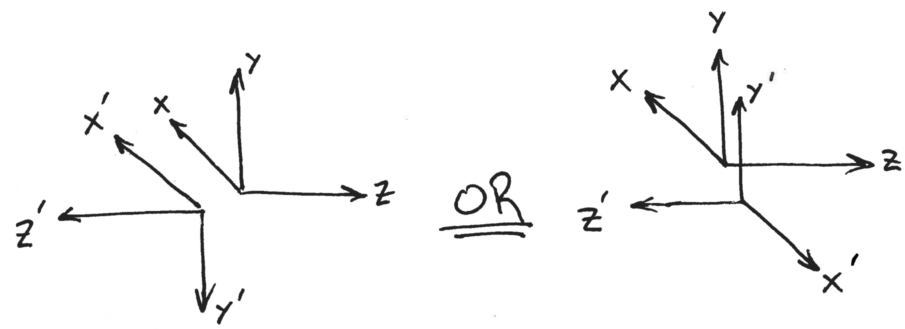
- Some materials are chiral and exhibit circular dichroism/birefringence
  → index of refraction different for right and left circ. polarized light
- materials with CD can be used to make <u>optical rotators</u>
  - input beam is rotated by an angle $\theta = dC$, where d is the distance propagated and C is the strength of optical activity (radians/meter)
- Since result is an active rotation of polarization, the Jones matrix is:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \underset{\text{(Polarization angle)}}{\theta_E} \longrightarrow \theta_E + \underset{\text{(rotation angle)}}{\theta_R}$$

Reflection

- to correctly represent a reflection, have to decide on how to map onto a new coordinate system after reflection

- to make a new right-handed coordinate system (after reflection) with $\hat{z} \rightarrow -\hat{z}$, also have to revise definition of either $\hat{x}$ or $\hat{y}$, but <u>not</u> both



- either choice is okay as long as you're being consistent

- for Fresnel reflection/transmission at some angle θ, you still have to calculate the coeff.'s by hand

$$\begin{bmatrix} -r_p(\theta) & 0 \\ 0 & r_s(\theta) \end{bmatrix} \quad \& \quad \begin{bmatrix} t_p(\theta) & 0 \\ 0 & t_s(\theta) \end{bmatrix}$$

phase shift due to reflection (assumed $\hat{x}' = -\hat{x}$ here)

where we've defined $\hat{x}$ along $s(\perp)$ & $\hat{y}$ along $p(\parallel)$ components