Physics 47 — Optics

Exam II

Due by 5pm on Wed, Nov 1, 2017

(give to me directly – Wilder 002)

- Allowed items: any resources necessary for solving the problems, **except** other students. *All solutions must be an individual effort!*
- Show all work in detail for full credit. Give best guess towards solution for partial credit.
- Use a separate sheet of paper for solution to each problem.
- Write your name on every sheet of solutions.
- When handing in, verify the order of solutions.

Problem 1: Phase and Group Velocity

(a) There are several variety of waves possible for large bodies of water. For instance, long-wavelength waves which extend deep into the ocean are dominated by Earth's gravitational acceleration, g, leading to a phase velocity:

$$v_p \approx \left(\frac{g\lambda}{2\pi}\right)^{\frac{1}{2}}.$$

On the other hand, short-wavelength waves at the surface have a phase velocity that is primarily determined by the density, ρ, and surface tension, T, of the water:

$$v_p \approx \left(\frac{2\pi T}{\lambda\rho}\right)^{\frac{1}{2}}.$$

Find the group velocity for each case as some factor of the corresponding phase velocity.

(b) Wave functions are also used to represent particles with mass, m, in quantum mechanics. For a free particle traveling in the $+\hat{\mathbf{x}}$ direction, having momentum, p, and energy, $E = p^2/2m$, the wave function is given as:

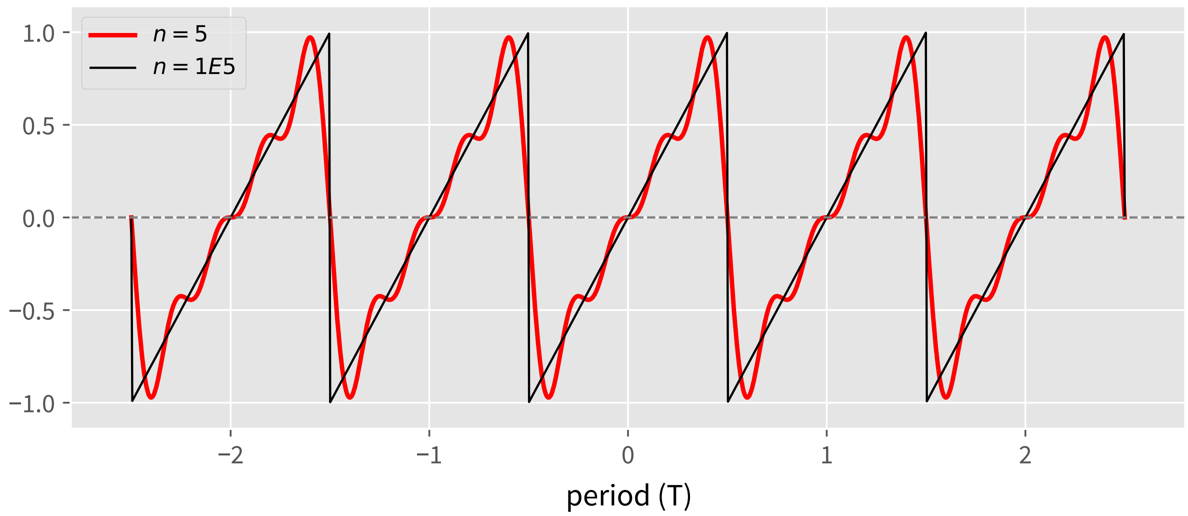
$$\Psi(x,t) = Ae^{i(px-Et)/\hbar}.$$

Calculate the phase and group velocities for this particle and tell which one corresponds to the classical speed of travel. (Note: the wave velocity is *half* the group velocity for a free particle.)

Problem 2: Fourier Series and Transform

Developing a mental faculty for the conversion of time-domain signals to frequency-domain (or visa-versa) is an invaluable skill to both the physicist and engineer. As an effort to stimulate this aptitude, consider the following (extremely) common functions.

(a) Find the Fourier series representation for the function $f(t) = \frac{2}{T}t$, defined over the period $-T/2 < t < T/2$.



How many coefficients do you think are necessary to faithfully reproduce the analytic result to within a 1% error? (This may be easiest to answer computationally, in which case the analytic solution can be approximated by $n = 1E5$ coefficients.)

(b) Find the Fourier transform, $\widetilde{f}(\omega)$, for the short sinusoidal pulse

$$f(t) = \begin{cases} sin(\omega_0 t) & -T/2 < t < T/2 \\ 0 & \text{otherwise.} \end{cases}$$

Plot or draw the original function, $f(t)$, over some span, $0 \leq t < t'$. Do the same for the Fourier transform magnitude, $|\widetilde{f}(\omega)|$, but over some span $-\omega' < \omega < \omega'$. Describe what happens to the Fourier transform as $T \to \infty$.

(c) Find the Fourier transform, $\widetilde{f}(\omega)$, for the exponential decay function (where $a > 0$)

$$f(t) = \begin{cases} 0 & t < 0 \\ e^{-at} & t \geq 0. \end{cases}$$

What is the magnitude, $|\widetilde{f}(\omega)|$? Plot or draw the original function, $f(t)$, over some span, $0 \leq t < t'$. Do the same for the Fourier transform magnitude, $|\widetilde{f}(\omega)|$, but over some span $-\omega' < \omega < \omega'$.

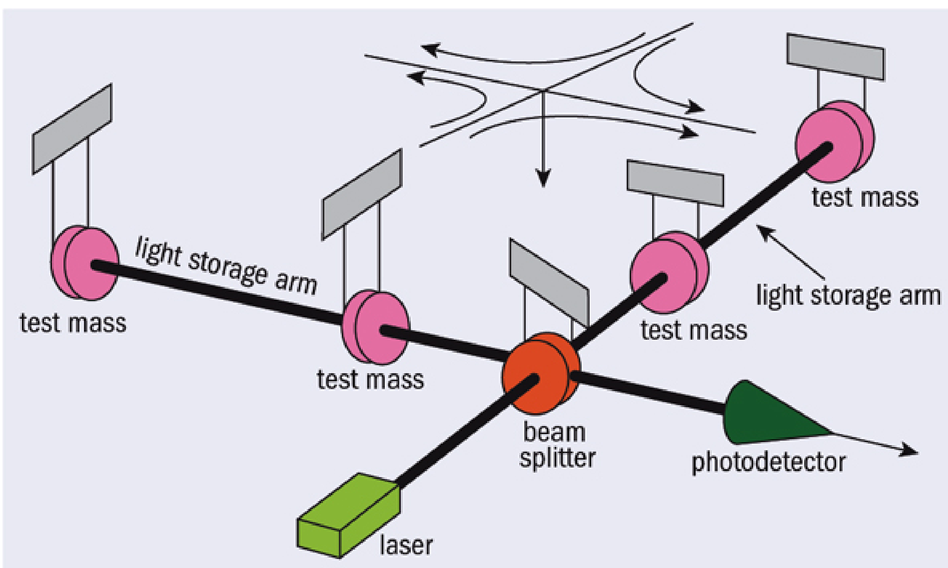
(d) Find the Fourier transform, $\widetilde{f}(\omega)$, for the exponential-damped sinusoid function (where $\sigma > 0$)

$$f(t) = \begin{cases} 0 & t < 0 \\ e^{-\sigma t} sin(\omega_0 t) & t \geq 0. \end{cases}$$

What is the magnitude, $|\widetilde{f}(\omega)|$? Plot or draw the original function, $f(t)$, over some span, $0 \leq t < t'$. Do the same for the Fourier transform magnitude, $|\widetilde{f}(\omega)|$, but over some span $-\omega' < \omega < \omega'$.

Problem 3: Gravitational Wave Detection at LIGO

The Laser Interferometric Gravitational Wave Observatory (LIGO) is a *large* Michelson interferometer, with Fabry-Perot cavities in each arm. The arm length is $L = 4$ km, the interferometer is in vacuum (n=1) and the light makes an average of 50 round-trips in the Fabry-Perot cavities in each arm. A gravitational wave from a merger of two nearby neutron stars should create a strain of $h = \delta L/L = 10^{-21}$ in the arms of a Michelson-type gravitational wave detector ($\delta L = L1 - L2$, the difference between the length of the two arms). Before the wave arrives, the interferometer is set to the "null" position, where no light is detected. You may neglect power losses due to absorption or scattering in the calculations below.



(a) Calculate the phase difference of the light ($\lambda = 488$ nm) arriving at the detector from the two arms of the interferometer due to this gravitational strain.

(b) Use this result to estimate the power exiting the interferometer toward the photodetector shown, if the laser output power P_0 is 10 W, and determine how many photons per second this corresponds to (Note: this will be a very small, but detectable quantity. The real challenge with this setup is distinguishing the signal from background noise.)

(c) On 17 August, 2017, LIGO observed the gravitational waves produced from the collision-merger of two neutron stars in a galaxy (NGC 4993) some 130×10^6 light-years away in the direction of the constellation Hydra. Such "kilonova" events have also been a suspected source of extremely short, yet extremely intense, gamma-ray bursts. Fortunately, the Fermi gamma-ray space telescope was able to simultaneously observe this particular kilonova event. Of great interest is the fact that these two detection methods, one of gravitational waves and one of electromagnetic waves (both massless), show a timing discrepancy for the arrival times of waves from this event – the gamma-ray photons were detected ~ 1.7 seconds *after* the gravitational waves!

Give some speculation as to the origin of this variance in terms of *effective* index of refraction.

Problem 4: Polarization and Optical Activity

In class, we unfortunately had little time to spend on the origins of optical activity due to circular birefringence and the utility of this effect in constructing polarization rotators. This is a fascinating topic with (quite literally) broad implications. So for this portion of the exam, I'll ask you to write an essay on the topic, focusing on the Faraday effect and including the following information:

- Qualitatively describe Faraday rotation and its origin.
- Derive Faraday rotation. In particular, develop a relation for $(n_{RC} - n_{LC})$ that depends on applied magnetic field, B_0 (or, alternatively, the electron cyclotron frequency, Ω_{cyc}).
- Discuss how this difference in indices leads to a rotation angle for a polarized EM field propagating over some distance L. Give an expression (quoting one is fine).
- What is the Verdet constant? Describe it and give an expression related to rotation angle (again, quoting this is fine, although following Becquerel's classical derivation is not too difficult).
- What is 'rotation measure'? Describe it and give an expression related to rotation angle (quoting one is fine).
- Faraday rotation has importance well beyond the realm of a traditional optics bench. Describe some application for Faraday rotation which is relevant to astronomy.

Obviously, neither the textbook or in-class materials covers this topic to this depth. You will therefore need to *exercise your resourcefulness!* Admittedly, this can become a very deep topic – feel free to avoid any resource involving tensors or quantum mechanics.