

(1)

1

(a)
$$v_p = \frac{d\omega}{dk} = \frac{d}{dk}(kv_p) = v_p + k \frac{dv_p}{dk}$$

where $k = \frac{2\pi}{\lambda}$ & $dk = -\frac{2\pi d\lambda}{\lambda^2}$

So
$$v_g = v_p + \frac{2\pi}{\lambda} dv_p \cdot \left(-\frac{1}{2\pi}\right) \frac{\lambda^2}{d\lambda} = v_p - \lambda \frac{dv_p}{d\lambda}$$

long λ :
$$\frac{dv_p}{d\lambda} = \frac{d}{d\lambda} \left(\frac{g\lambda}{2\pi}\right)^{\frac{1}{2}} = \frac{1}{2} \sqrt{\frac{g}{2\pi\lambda}}$$

$$v_g = v_p - \frac{\lambda}{2} \sqrt{\frac{g}{2\pi\lambda}} = v_p - \frac{1}{2} v_p = \underline{\underline{\frac{1}{2} v_p}}$$

short λ :
$$\frac{dv_p}{d\lambda} = \frac{d}{d\lambda} \left(\frac{2\pi T}{\lambda \rho}\right)^{\frac{1}{2}} = \frac{-1}{2\lambda} \left(\frac{2\pi T}{\lambda \rho}\right) = -\frac{1}{2\lambda} v_p$$

$$v_g = v_p - \lambda \left(-\frac{1}{2\lambda} v_p\right) = \underline{\underline{\frac{3}{2} v_p}}$$

(b) basically have $\frac{i}{\hbar}(px - Et) = i(kx - \omega t) \Rightarrow k = \frac{p}{\hbar}$ & $\omega = \frac{E}{\hbar}$

but $\omega = \frac{E}{\hbar} = \frac{p^2}{2m\hbar}$, so $v_p = \frac{\omega}{k} = \frac{E}{p} = \frac{p}{2m} = \underline{\underline{\frac{\hbar k}{2m}}}$

and $v_g = \frac{d\omega}{dk} = \frac{2\hbar k}{2m} = \underline{\underline{\frac{\hbar k}{m} = \frac{p}{m_1}}}$, so $v_p = \frac{1}{2} v_g$

Since classical momentum is $p = mv$, then v_g here is classical velocity

(a) Complex Fourier coeff's are:

$$F_n = \frac{1}{T} \int_{-T/2}^{T/2} \frac{2}{T} t e^{in\omega t} dt$$

$$= \frac{2i}{n\omega T^2} \left[t e^{-in\omega t} \right]_{-T/2}^{T/2} + \int_{T/2}^{T/2} \frac{1}{in\omega} e^{-in\omega t} dt$$

$$= \frac{i}{2n\omega} \left[e^{-in\pi} + e^{in\pi} \right] + 0$$

$$= \frac{i}{n\pi} \cos(n\pi)$$

$$= \frac{i(-1)^n}{n\pi} \quad [n \neq 0]$$

since $\cos(n\pi) = (-1)^n$

With a zero freq. component

$$F_0 = \frac{1}{T} \int_{-T/2}^{T/2} \frac{2}{T} t dt = 0$$

So the Fourier Series is

$$f(t) = \sum_{n=1}^{\infty} \frac{i(-1)^n}{n\pi} \left(e^{in\omega t} - e^{-in\omega t} \right)$$

$$= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\omega t)$$

[see computational solutions]²

2

$$\begin{aligned}
\textcircled{b} \quad \tilde{f}(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \\
&= \int_{-T/2}^{T/2} \sin(\omega_0 t) e^{-i\omega t} dt \\
&= \frac{1}{i2} \int_{-T/2}^{T/2} \left[e^{-i(\omega-\omega_0)t} - e^{-i(\omega+\omega_0)t} \right] dt \\
&= \frac{1}{2(\omega-\omega_0)} \left[e^{-i(\omega-\omega_0)t} \right]_{-T/2}^{T/2} - \frac{1}{2(\omega+\omega_0)} \left[e^{-i(\omega+\omega_0)t} \right]_{-T/2}^{T/2} \\
&= \frac{-iT}{2} \left[\frac{\sin\left(\frac{T}{2}(\omega-\omega_0)\right)}{\frac{T}{2}(\omega-\omega_0)} - \frac{\sin\left(\frac{T}{2}(\omega+\omega_0)\right)}{\frac{T}{2}(\omega+\omega_0)} \right] \\
&= \frac{-iT}{2} \left[\text{sinc}\left(\frac{T}{2}(\omega-\omega_0)\right) - \text{sinc}\left(\frac{T}{2}(\omega+\omega_0)\right) \right]
\end{aligned}$$

$$\begin{aligned}
\textcircled{c} \quad \tilde{f}(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \\
&= \int_0^{\infty} e^{-at} e^{-i\omega t} dt \\
&= \left[\frac{-1}{a+i\omega} e^{-(a+i\omega)t} \right]_0^{\infty} \\
&= \frac{1}{a+i\omega}
\end{aligned}$$

$$\begin{aligned}
|\tilde{f}(\omega)| &= \sqrt{\tilde{f} \cdot \tilde{f}^*} \\
&= \left[\frac{1}{a+i\omega} \cdot \frac{1}{a-i\omega} \right]^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{a^2 + \omega^2}}
\end{aligned}$$

[Plots shown in computational solutions]

2
d

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$= \frac{1}{i2} \int_0^{\infty} e^{-\sigma t} (e^{i\omega_0 t} - e^{-i\omega_0 t}) e^{-i\omega t} dt$$

$$= \frac{1}{i2} \left[\frac{-1}{\sigma + i(\omega_0 - \omega)} e^{-(\sigma + i(\omega_0 - \omega))t} \right]_0^{\infty}$$

$$- \frac{1}{i2} \left[\frac{-1}{\sigma + i(\omega_0 + \omega)} e^{-(\sigma + i(\omega_0 + \omega))t} \right]_0^{\infty}$$

$$= \frac{\omega_0}{(\sigma + i\omega)^2 + \omega_0^2}$$

$$|\tilde{f}(\omega)| = \sqrt{\tilde{f} \cdot \tilde{f}^*}$$

$$= \frac{\omega_0^2}{(\sigma + i\omega)^2 + \omega_0^2} \cdot \frac{\omega_0^2}{(\sigma - i\omega)^2 + \omega_0^2}$$

$$= \frac{\omega_0}{\sqrt{(\sigma^2 + \omega_0^2 - \omega^2)^2 + (2\sigma\omega)^2}}$$

[plot shown in computational solutions]

P47 - Exam II - Solutions

November 1, 2017

1 Exam II Solutions

```
In [1]: from math import *
import numpy as np

import matplotlib.pyplot as plt
# plt.style.use('fivethirtyeight')
plt.style.use('ggplot')
%matplotlib inline

In [2]: def mpl_fig_template(fig_handle, title, x_ax_label, y_ax_label):

    from matplotlib import font_manager

    fig_handle.set_size_inches(9,3.5)
    fig_handle.dpi = 120

    # font = 'Letter Gothic Std'
    font = 'Source Sans Pro'

    if title != None:
        plt.title(title, fontname='DejaVu Sans', fontsize=10, loc='left')

    plt.xlabel(x_ax_label, fontname=font, fontsize=14)
    plt.ylabel(y_ax_label, fontname=font, fontsize=14)

    ticks_font = font_manager.FontProperties(family=font, size=12)

    ax = plt.gca()

    x_ax_lim = ax.get_xbound()
    y_ax_lim = ax.get_ybound()

    x_extend = ((x_ax_lim[1]- x_ax_lim[0]))*0.01
    y_extend = ((y_ax_lim[1]- y_ax_lim[0]))*0.02

    ax.set_xbound(x_ax_lim[0] - x_extend, x_ax_lim[1] + x_extend)
```

```

ax.set_ybound(y_ax_lim[0] - y_extend, y_ax_lim[1] + y_extend)

for label in ax.get_xticklabels():
    label.set_fontproperties(ticks_font)

for label in ax.get_yticklabels():
    label.set_fontproperties(ticks_font)

def mpl_label_template(coords, label_list):

    x_perc, y_perc = coords
    ax = plt.gca()

    label = '\n'.join(label_list)
    font = 'Letter Gothic Std'

    ax.text(x_perc, y_perc, label, transform=ax.transAxes, fontname=font,
            fontsize=10.5, bbox={'facecolor':'white', 'alpha':0.65, 'pad':5})

```

2 Problem 2(a)

```

In [3]: def f(w, t, n):
        order_coeffs = 0
        for m in np.arange(n)+1:
            order_coeffs += (-1)**(m+1)/m * np.sin(m*w*t)
        return 2*order_coeffs/np.pi

In [4]: w = 2*np.pi
        t = np.linspace(-2.5, 2.5, 1000)
        n = 5

        f_n = f(w, t, n)
        f_act = f(w, t, 1E5)

        title = r''
        x_ax_label = r'period(T)'
        y_ax_label = ''

        fig_fn = plt.figure()

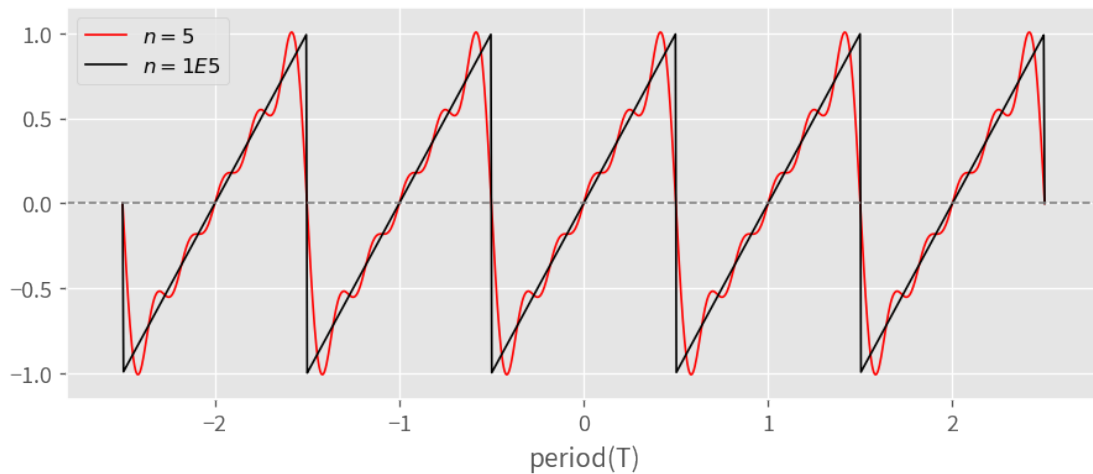
        plt.plot(t, f_n, color='r', linewidth=1, label=r'$n=5$')
        plt.plot(t, f_act, color='k', linewidth=1, label=r'$n=1E5$')

        mpl_fig_template(fig_fn, title, x_ax_label, y_ax_label)
        plt.legend()

        plt.axhline(y=0.0, color='grey', linestyle='--', linewidth=1)

```

```
plt.savefig('sawtooth.pdf', dpi=120, format='pdf')
plt.show()
```



```
In [5]: n = 1
        target_err = 1.0
        actual_err = 100
        t = np.linspace(-0.5, 0.5, 1000)

        f_n = f(w, t, n)
        f_act = f(w, t, 1E5)

        while actual_err > target_err:
            n += 1
            f_n = f(w, t, n)
            actual_err = 100*np.abs(np.mean(1 - f_n/f_act))

        print('n =', n, '\n% err =', actual_err)
```

```
n = 27
% err = 0.97901190208
```

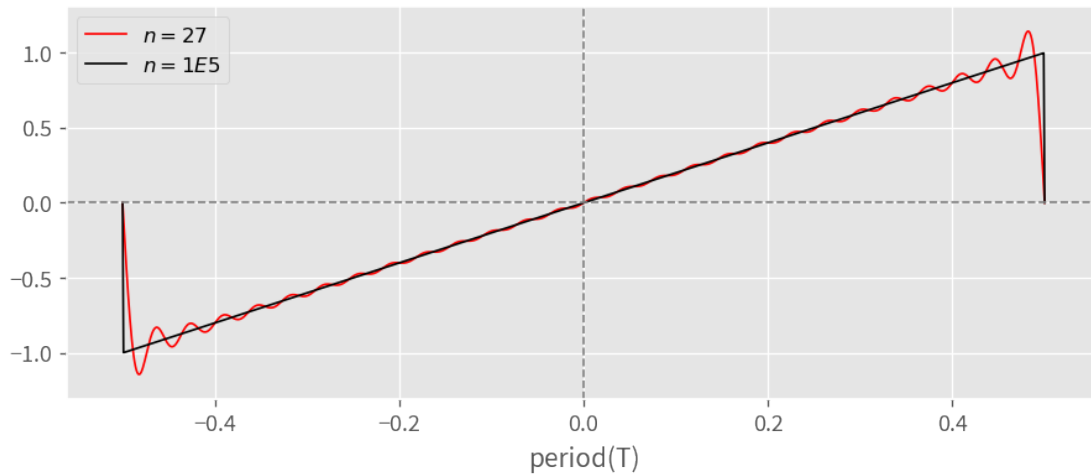
```
In [6]: fig_fn = plt.figure()

        plt.plot(t, f_n, color='r', linewidth=1, label=r'$n={}$'.format(n))
        plt.plot(t, f_act, color='k', linewidth=1, label=r'$n=1E5$')

        mpl_fig_template(fig_fn, title, x_ax_label, y_ax_label)
        plt.legend()

        plt.axhline(y=0.0, color='grey', linestyle='--', linewidth=1)
```

```
plt.axvline(x=0.0, color='grey', linestyle='--', linewidth=1)
# plt.savefig('sawtooth_soln.pdf', dpi=120, format='pdf')
plt.show()
```



3 Problem 2(b)

```
In [7]: def f2(w, w0, T):
        return -1j*T*(np.sinc((w-w0)*T/2) - np.sinc((w+w0)*T/2))/2
```

```
In [8]: w0 = 1
        w = np.linspace(-5*w0, 5*w0, 1000)

        T = 10
        fn_2 = np.absolute(f2(w, w0, T))

        T = 100
        fn_2b = np.absolute(f2(w, w0, T))

        x_ax_label = r'\omega$'
        y_ax_label = 'normalized'

        fig_fn2 = plt.figure()

        plt.plot(w, fn_2/np.max(fn_2), color='r', linewidth=1, label=r'$T=10$')
        plt.plot(w, fn_2b/np.max(fn_2b), color='k', linewidth=1, label=r'$T=100$')

        mpl_fig_template(fig_fn2, title, x_ax_label, y_ax_label)
        plt.legend()

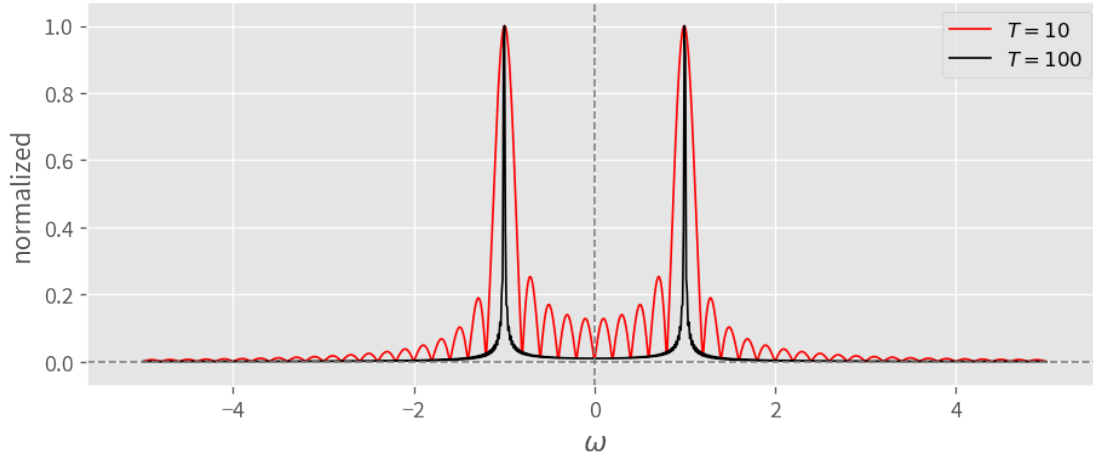
        plt.axhline(y=0.0, color='grey', linestyle='--', linewidth=1)
```



```

plt.axvline(x=0.0, color='grey', linestyle='--', linewidth=1)
# plt.savefig('sin_pulse.png', dpi=120, format='png')
# plt.savefig('sin_pulse.pdf', dpi=120, format='pdf')
plt.show()

```



4 Problem 2(c)

```

In [9]: def f3(a, w):
        return 1/(a+1j*w)

```

```

In [10]: a = 5
        w = np.linspace(-25*a, 25*a, 1000)

        fn_3 = np.absolute(f3(a, w))

        fig_fn3 = plt.figure()

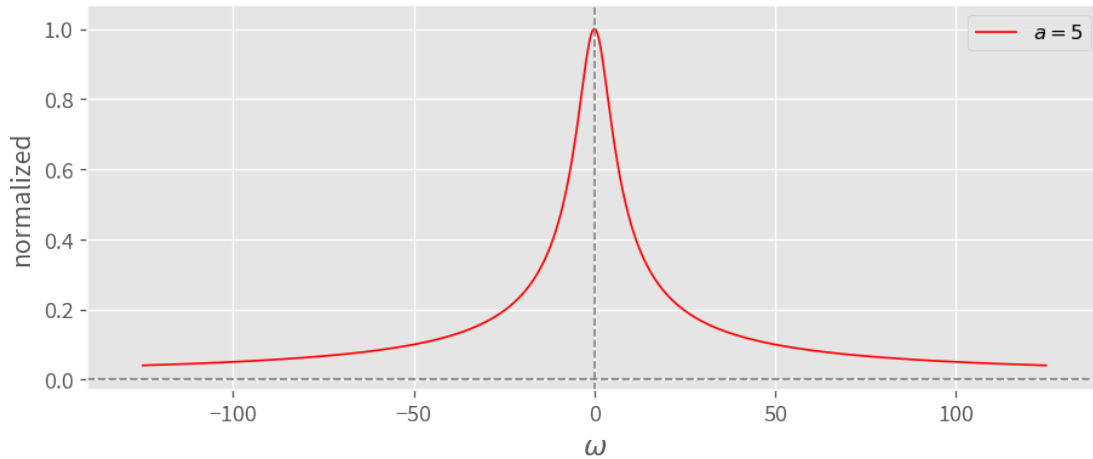
        plt.plot(w, fn_3/np.max(fn_3), color='r', linewidth=1, label=r'$a=5$')

        mpl_fig_template(fig_fn3, title, x_ax_label, y_ax_label)
        plt.legend()

        plt.axhline(y=0.0, color='grey', linestyle='--', linewidth=1)
        plt.axvline(x=0.0, color='grey', linestyle='--', linewidth=1)

        plt.show()

```



5 Problem 2(d)

```
In [11]: def f4(sigma, w, w0):
         return w0/((s + 1j*w)**2 + w0**2)
```

```
In [12]: s = 0.25
         w0 = 10
         w = np.linspace(-3*w0, 3*w0, 1000)

         fn_4 = np.absolute(f4(s, w, w0))

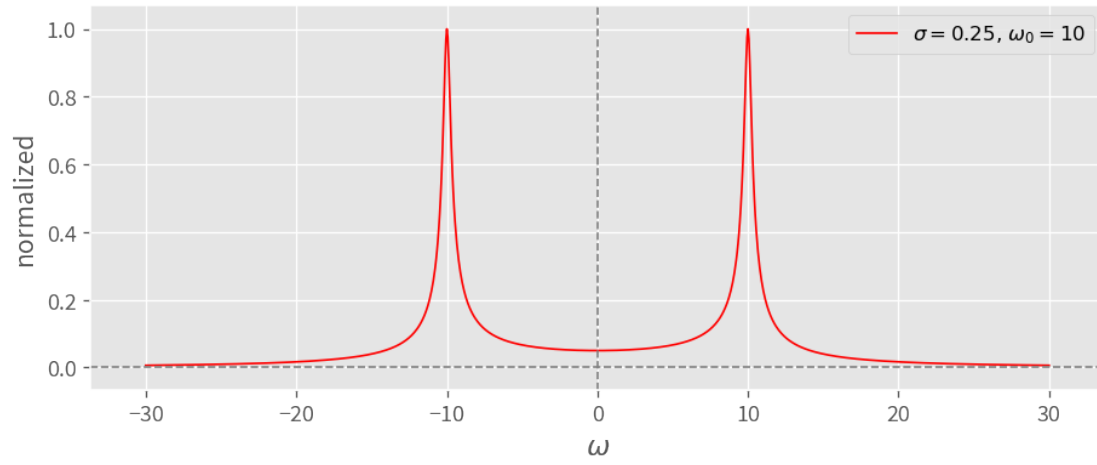
         fig_fn4 = plt.figure()

         plt.plot(w, fn_4/np.max(fn_4), color='r', linewidth=1, label=r'\sigma=0.25, \omega')

         mpl_fig_template(fig_fn4, title, x_ax_label, y_ax_label)
         plt.legend()

         plt.axhline(y=0.0, color='grey', linestyle='--', linewidth=1)
         plt.axvline(x=0.0, color='grey', linestyle='--', linewidth=1)

         plt.show()
```



3

(a) Power at the detector is given by

$$I_{\text{det}} = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \delta$$

$$P_{\text{det}} = P_1 + P_2 + 2\sqrt{P_1 P_2} \cos \delta$$

Since the beam splitter is 50/50, then

$$P_{\text{det}} = \frac{P_0}{4} + \frac{P_0}{4} + 2 \frac{P_0}{4} \cos \delta = \frac{P_0}{2} (1 + \cos \delta)$$

At the "null" condition, power at detector has to be zero, so the phase shift has to be

$$\delta = \pi + \delta_g \quad (\delta_g : \text{phase shift due to gravitational wave})$$

$$\Rightarrow P_{\text{det}} = \frac{P_0}{2} (1 + \cos \delta) = \frac{P_0}{2} (1 + \cos(\pi + \delta_g)) = \frac{P_0}{2} (1 - \cos \delta_g)$$

$$\Rightarrow P_{\text{det}} = P_0 \sin^2(\delta_g/2), \quad \text{using half-angle identity}$$

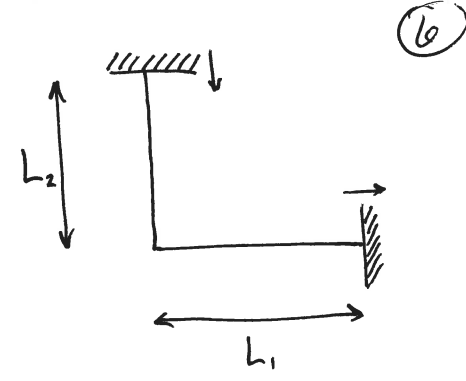
$$\text{and since } \delta_g \text{ is small, } P_{\text{det}} \approx P_0 (\delta_g/2)^2$$

At the same time, the phase difference induced by the gravitational wave is going to be $k\Delta s$, where s is the difference in path length traveled by the beams in each arm of the interferometer

3
a-cont.

light travels length of arm twice

Passes due to Fabry-Perot



$$\Delta S = 2 \cdot 50 \cdot \Delta L = 100 h L$$

$$\Delta L = L_1 - L_2 = h L \Rightarrow h = \frac{L_1 - L_2}{L}$$

means the phase shift caused by gravity strain

$$\begin{aligned} \delta_g &= k \Delta S = \frac{2\pi}{\lambda} \cdot 100 h L \\ &= \frac{200\pi}{(488 \text{ nm})} (10^{-21}) (4 \text{ km}) \\ &= \underline{\underline{5.15 \times 10^{-9} \text{ radians}}} \end{aligned}$$

b) Power at the detector is then

$$P_{\text{det}} = P_0 \left(\frac{\delta_g}{2} \right)^2 = (10 \text{ W}) \left(\frac{5.15 \times 10^{-9}}{2} \right)^2 = \underline{\underline{6.63 \times 10^{-17} \text{ W}}}$$

Number of photons per second is going to be

$$N_g = \frac{P_{\text{det}}}{E_g} = \frac{P_{\text{det}}}{hc/\lambda} = \frac{(6.63 \times 10^{-17} \text{ W})}{(4.07 \times 10^{-19} \text{ J})} = \underline{\underline{163 \frac{\text{photons}}{\text{sec.}}}}$$

c) [See Next Page]

Problem 3(c): Gravitational Wave Detection at LIGO

On 17 August, 2017, LIGO observed the gravitational waves produced from the collision-merger of two neutron stars in a galaxy (NGC 4993) some 130×10^6 light-years away in the direction of the constellation Hydra. Such “kilonova” events have also been a suspected source of extremely short, yet extremely intense, gamma-ray bursts. Fortunately, the Fermi gamma-ray space telescope was able to simultaneously observe this particular kilonova event. Of great interest is the fact that these two detection methods, one of gravitational waves and one of electromagnetic waves (both massless), show a timing discrepancy for the arrival times of waves from this event – the gamma-ray photons were detected ~ 1.7 seconds *after* the gravitational waves!

Give some speculation as to the origin of this variance in terms of *effective* index of refraction.

This recent observation turns out to be an *extremely* impactful result as it directly impacts and limits the range of possible theories used to describe dark matter. To begin with, remember that the universal speed limit in our universe is that of light in an empty vacuum (no fields, no matter, etc.). However, as we’ve seen throughout this course, the velocity of light may be somewhat damped via interactions of the electromagnetic field with matter. The context on which we’ve focused our attention in particular has been between EM waves and local dipole moments of dielectric materials. But there is yet another manner in which light may effectively couple to matter, and that is through the gravitational force’s distortion of the spacetime metric itself – in the presence of matter (i.e. gravitational potential), photons no longer follow a straight-line path, but instead curve around dense objects according to the curvature of the local spacetime. So, instead of a travel time determined purely by the vacuum speed of light and a straight-line distance between source and observer, if there are appreciable gravitational potentials in the intervening space, the path length is effectively increased. This results in a time delay from what would be expected for the straight-line path, where this delay is termed the Shapiro delay.

Back to point, the difference in Shapiro delay for the EM waves and gravity waves is very telling of how gravity itself couples to matter. Of particular interest is how gravity couples to *dark* matter since this enables inference of the nature of dark matter. We have many astronomical observations for how light is affected by the presence of dark matter, but have been relatively blind in this regard for gravity itself.

Two major theoretical camps in the gravity/dark matter community basically have this to say about dark matter: 1) There *is* some form of (particle-based) dark matter and it primarily interacts with gravity, but not so much to EM fields or regular matter; 2) There is *no* dark matter, it’s just that the nature of gravity itself must be modified on galactic length scales, which makes it *seem* like there is some unobserved mass that should be present (the so-called MODified Newtonian Gravity, or MOND, theory). Each camp has had its triumphs

and challenges, so it remains unclear which view is actually correct. That is, until now... (sort of.)

You see, if gravitational waves were to travel directly from NGC 4993 to us at roughly the vacuum speed of light, experiencing marginal Shapiro delay from the “dark matter” which afflicts electromagnetic waves, then the gravitational wave would arrive to us approximately *1000 days before* the gamma-ray burst! And this is definitely not the case! Instead, it seems that each form of radiation experiences nearly the same path-length difference, strengthening the case for a dark matter composed of actual particle-based matter. This isn’t to say that MOND is dead – it’s a fairly flexible theoretical framework and can possibly be retooled to fit this observation. It is, however, severely constrained, which is really the most we can ask for in physics.

If you’d like to learn more about this, do check out the recent arXiv paper by S. Boran, et al: [*GW170817 Falsifies Dark Matter Emulators*]

In any case, for the problem at hand, you can contextualize the individual Shapiro delays through an effective index of refraction for each type of field; n_γ for EM waves and n_g for gravity waves. The difference in Shapiro delay can then be folded into $\Delta n = n_\gamma - n_g$, where

$$\Delta n = n_\gamma - n_g = \frac{c}{v_\gamma} - \frac{c}{v_g}.$$

To consider the effects of Shapiro delay, you can (naively) consider that either time of travel is affected *or* distance of travel is affected (since space and time are on equal footing in the ‘spacetime’). For our purposes, let’s assume distance traveled is fixed at $d = 130 \times 10^6$ light-years. Then, given the timing delay of 1.7 seconds,

$$\Delta n = c \frac{t_\gamma}{d} - \frac{t_g}{d} = \frac{c}{d} |t_\gamma - t_g|$$

$$\Delta n = 4.15 \times 10^{-16},$$

which is absolutely tiny, but nonetheless observable over intergalactic distances for these two wave types. The fact that it’s tiny, though, is what is so remarkable.

Problem 4: Polarization and Optical Activity

[Credit: The following pages are from the exam solutions of Raphael Hviding]

Problem 4: Polarization and Optical Activity

Faraday rotation is the result of an interaction between an electromagnetic wave and a background magnetic field that results in the rotation of the angle of polarization of the incident wave. In general, this is most commonly observed in nature with radio waves from extraterrestrial sources (extrasolar, in fact). A predominant radio source in the cosmos is Cyclotron radiation, which occurs from the acceleration of charges in a magnetic field. In general, Cyclotron radiation is polarized. In order for astronomers to properly understand the polarization of the source material, they must take into account the effect of Faraday rotation for intervening magnetic fields. This is one motivation for understanding Faraday rotation.

Intuitively, how can we understand Faraday rotation? Imagine an electromagnetic wave propagating in the same direction as some electric field. Charged particles in this field will circle the magnetic field lines, with the direction depending on their charge. The moving charged particles will additionally create their own magnetic field, in addition to the external one. In general, we can express a linearly polarized wave as the sum of two circularly polarized waves. Now, the extra magnetic fields created will increase one of the two (circularly polarized) electric field components, while the other component will be decreased. Now, this corresponds to an effective phase difference between the two components, which leads to an effective rotation of the polarization of the final wave after exiting the area of magnetic field activity.

“Where’s the math?” you ask. Never fear, following is a derivation of Faraday rotation for Cyclotron radiation:

First, let’s recognize that we are only dealing with electrons, the force on which can be described by the vector equation:

$$\vec{F} = -e \left(\vec{E} + \frac{1}{c} \vec{v} \times \vec{B} \right)$$

Now, again from our intuitive description, imagine the magnetic field and the direction of wave propagation are parallel. Now, we relate the natural oscillatory motion of the ambient particles to the force described above.

$$m \frac{d^2 \vec{s}}{dt^2} + \omega_0^2 m \vec{s} = -e \left(\vec{E} + \frac{1}{c} \frac{d\vec{s}}{dt} \times \vec{B} \right)$$

where \vec{s} is the dipole motion, ω_0 is the natural frequency of oscillation, and m is, as always, the mass.

Now of course, the electromagnetic wave will have a different frequency, namely ω , than the natural frequency. Now because of this, at equilibrium each component will have an $\exp(-i\omega t)$ dependence. Now, for convenience, we separate the x and y components. and retrieve the following expressions.

$$(\omega_0 - \omega)s_x - i \frac{eB}{mc} \omega s_y = -\frac{e}{m} E_x, \quad (\omega_0 - \omega)s_y + i \frac{eB}{mc} \omega s_x = -\frac{e}{m} E_x$$

These come from differentiating \vec{s} assuming it has this dependence. The cyclotron frequency is given by $\frac{eB}{mc}$ and is denoted Ω . Now, we want to retrieve the expressions for circularly polarized light that define our linearly polarized fields. We can construct linear polarization by the following:

$$E_x = \frac{1}{2}(E_+ + E_-), \quad E_y = \frac{1}{2i}(E_+ - E_-),$$

Where E_{\pm} are the right and left hand polarized waves respectively. Now, it's just a little algebra to show that:

$$E_{\pm} = E_x \pm iE_y \text{ and we do the same for } s_{\pm} = s_x \pm is_y$$

Now, substituting into the earlier equation we arrive at:

$$(\omega_0^2 - \omega^2 - \omega\Omega)s_+ = -\frac{e}{m}E_+$$

$$(\omega_0^2 - \omega^2 + \omega\Omega)s_- = -\frac{e}{m}E_-$$

Now the induced dipole moment is given by $P = n_e es$, where n_e is the number density of electrons, which we can recast with the above:

$$\Rightarrow P_{\pm} = \frac{n_e e^2 E_{\pm}}{(\omega_0^2 - \omega^2 \mp \omega\Omega)s_{\pm}}$$

Now, we can find the polarizability of the material, given by $\chi = 4\pi P/E$ and then the index of refraction, given by $n = \sqrt{1 + \chi}$. So, recasting these with the above expressions:

$$n_{\pm} = \sqrt{1 + \frac{4\pi n_e e^2}{m(\omega_0^2 - \omega^2 \mp \omega\Omega)}}$$

Clearly $n_+ > n_-$, which implies the effective velocity of the right handed wave is faster than the left handed wave.

Now, solving for the new x and y polarized waves:

$$E_x = E_0 \cos\left(\frac{\omega}{c}[n_+z - ct]\right) + E_0 \cos\left(\frac{\omega}{c}[n_-z - ct]\right)$$

$$E_y = E_0 \sin\left(\frac{\omega}{c}[n_+z - ct]\right) - E_0 \sin\left(\frac{\omega}{c}[n_-z - ct]\right)$$

Using trig identities, and letting $n = \frac{1}{2}(n_+ + n_-)$, we can rewrite these as:

$$E_x = 2E_0 \cos\left(\frac{\omega}{c}[nz - ct]\right) \cos\left(\frac{\omega}{2c}[n_+ - n_-]z\right)$$

$$E_y = 2E_0 \cos\left(\frac{\omega}{c}[nz - ct]\right) \sin\left(\frac{\omega}{2c}[n_+ - n_-]z\right)$$

Now the polarization angle of the new electric wave is given by $\tan(\varphi) = \frac{E_y}{E_x}$

$$\Rightarrow \varphi = \arctan\left(\frac{\sin\left(\frac{\omega}{2c}[n_+ - n_-]z\right)}{\cos\left(\frac{\omega}{2c}[n_+ - n_-]z\right)}\right) = \frac{\omega}{2c}[n_+ - n_-]z$$

Well, we want to characterize an expression for $n_+ - n_-$.

Now, in general, $\omega_0, \omega \gg \Omega$. So, in this limit, we can find n_{\pm} and we retrieve:

$$n_{\pm} \approx 1 \pm \frac{2\pi n_e e^2 \omega \Omega}{m(\omega_0^2 - \omega^2)^2}$$

$$\Rightarrow n_+ - n_- \approx \frac{4\pi n_e e^2 \omega \Omega}{m(\omega_0^2 - \omega^2)^2}$$

So $n_+ - n_-$ as no dependence on z , so:

$$\frac{d\varphi}{dz} = \frac{\omega}{2c}[n_+ - n_-] = \frac{\omega}{2c} \frac{4\pi n_e e^2 \omega \Omega}{m(\omega_0^2 - \omega^2)^2}$$

So in order to find the total angle change, we integrate with respect to z over the line of sight to the target.

$$\Delta\varphi = \frac{\omega}{2c}[n_+ - n_-] = \frac{2\pi e^3 \omega^2}{m^2 c^2 (\omega_0^2 - \omega^2)^2} \int_{\text{LOS}} n_e \frac{\Omega m c}{e} dz$$

Now, we consider the cases where $\omega \gg \omega_0$, since the radiation we are considering generally has higher frequencies than the natural oscillation of atoms.

$$\Rightarrow \Delta\varphi = \frac{\omega}{2c}[n_+ - n_-] = \frac{2\pi e^3}{m^2 c^2 \omega^2} \int_{\text{LOS}} n_e B dz$$

And now expression $\omega = \frac{2\pi c}{\lambda}$

$$\Rightarrow \Delta\varphi = \frac{\omega}{2c}[n_+ - n_-] = \frac{e^3 \lambda^2}{2\pi (mc^2)^2} \int_{\text{LOS}} n_e(z) B_z(z) dz$$

Now, it is usual to express this as a function of rotation measure (RM) $\Rightarrow \Delta\varphi = \text{RM} \lambda^2$

$$\Rightarrow \text{RM} = \frac{e^3}{2\pi (mc^2)^2} \int_{\text{LOS}} n_e(z) B_z(z) dz = (2.64 \times 10^{-17} \text{ Gauss}^{-1}) \int_{\text{LOS}} n_e(z) B_z(z) dz$$

where this describes the relationship between the rotation angle and wavelength for the same line of sight magnetic field and electron number density.

Now the Verdet constant, \mathcal{V} , describes the strength of the Faraday effect in a given material s.t.:

$$\Delta\varphi = \frac{e}{2mc} \omega \frac{dn}{d\omega} \int_{\text{LOS}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{z}} = \mathcal{V} \int_{\text{LOS}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{z}}$$

which arises from a different treatment of the proof. It highlights the importance of the dependence of the rotation angle on the derivative of the index of refraction with respect to the angular frequency of the radiation.

While I previously discussed the astronomical connection to Faraday radiation, it would be cool to give a specific example. One fascinating area of research, and a fantastic example of the ‘‘It’s Never Aliens’’ rule of thumb, are pulsars. Rapidly rotating neutron stars, the last remaining artifact of a massive star, is able to produce an inordinate amount of pulsed energy some of it in the form of highly polarized Cyclotron radiation, or it’s relativistic cousin, Synchrotron radiation. By measuring polarization angles across the radio spectrum, astronomers are able to extrapolate the rotation measure of the intervening space. This is done by assuming a common original polarization angle over a range of frequencies, but each is rotated by a different amount. Astronomers can therefore measure the integrated particle density times the magnetic field. With this technique,

we can understand the interstellar medium in the Milky Way, which is pretty cool! Go Faraday Rotation!

Sources:

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(https://www.physics.rutgers.edu/grad/506/Faraday_rotation.pdf)

University of Texas Notes on Faraday Rotation

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University of Michigan Notes on Faraday Optical Rotation

(Carl W. Akerlof, July 2009)

Wikipedia

(Pages on (1) Faraday Effect, (2) Cyclotron Radiation, (3) Verdet Constant)